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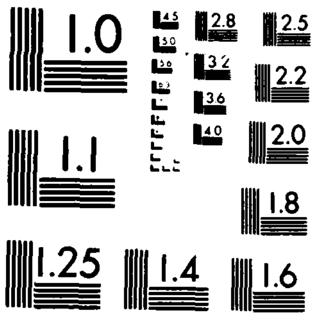

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DIFFERENTIAL ABSORPTION LIDAR: EFFECTS OF SPECKLE NOISE

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DIFFERENTIAL ABSORPTION LIDAR: EFFECTS OF SPECKLE NOISE

KJELL ÖSTBERG

1. INTRODUCTION

There is today a need for methods for remote monitoring in real time of air pollutants. Among the methods studied for this purpose during the last years, the differential absorption lidar (DIAL) technique has been found to be the most promising one (Kildal & Byer, 1971; Byer & Garbuny, 1973; Hinkley, 1976).

In the DIAL technique one uses a laser radar with a frequency which can be tuned in the neighborhood of an absorption line of a gas in the atmosphere. One measures the intensity of light backscattered from atmospheric aerosol particles, a topographical target, or a retroreflector. From the ratio of backscattered intensity at two wavelengths, one on and one off the particular absorption line, the gas content in the air column between the laser radar and the scattering volume can be obtained.

It is advantageous to make DIAL measurements in the infrared region of the spectrum. Most molecules have absorption lines in that region, and, furthermore, infrared lasers are eye-safe. However, the direct-detection technique, which has mostly been used for DIAL measurements so far, is comparatively insensitive in the infrared. The sensitivity of the DIAL measurement will be limited by the thermal noise in the detector. It was therefore suggested by Inaba & Kobayasi (1975) and Kobayasi & Inaba (1975) that one should instead use heterodyne detection in the DIAL scheme. Then it would ideally be possible to obtain quantum-limited operation, i.e., shot noise induced by the local oscillator would become the dominating noise source. They estimated that the DIAL sensitivity could be increased by several orders of magnitude. Later, Menzies & Shumate (1976) and Menzies (1978) made successful DIAL measurements with heterodyne detection of ozone, nitric oxide, and ethylene. In these measurements they used CW lasers, and the light was backscattered from a retroreflector or from rough surfaces.

With a pulsed lidar system and utilizing the light backscattered from atmospheric aerosol particles, it is possible to obtain range-resolved measurements. However, if heterodyne detection is used in this case, there will be large fluctuations from pulse to pulse in the received signal. The reason is that the backscattered electromagnetic fields from all the aerosol particles interfere with each other to set up the total backscattered field. In a time of the order of $\lambda/v \approx 1 \mu s$ (λ = wavelength, v = typical aerosol-particle speed), the particles have reshuffled so that a new sample of the random backscattered field is obtained. If the pulse time t_p is $<\lambda/v$, the coherence time instead is set by t_p, since after that time, there also is a new independent sample of the backscattered field. This phenomenon has been studied for a long time in connection with microwave radars (see, e.g., Marshall & Hitschfeld, 1953; Wallace, 1953). In optics these fluctuations in received intensity are named *speckle noise* (Dainty, 1975; Goodman, 1976). It is to be expected that this speckle noise will seriously degrade the sensitivity of a range-resolving DIAL system with heterodyne detection. The operation of the system will then no longer be quantum limited.

The problem with the speckle noise is in general much smaller by direct detection. The reason for this is that the backscattered field has a finite lateral coherence length. By heterodyne detection, reception takes place over only one coherence area. By direct detection, on the other hand, there usually is a spatial averaging over many coherence areas in the receiver aperture. Thus, the variance of the intensity fluctuations is correspondingly decreased in that case. Furthermore, averaging in the frequency domain also may take place in the direct-detection case.

The purpose of this report is to demonstrate, in more detail, the difference between direct detection and heterodyne detection with regard to the speckle noise. This is done in Chapters 2–4. Then, in Ch. 5, I consider the sensitivity limitations imposed by the speckle noise on a range-resolving differential absorption lidar with heterodyne detection.

2. DIRECT DETECTION

I shall derive an expression for the signal-to-noise ratio which takes into account the speckle noise. The incident light intensity is I. The detected power becomes

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$$P = \iint_A I\, d^2r,$$

where the integration is over the receiving aperture area A. The mean value of the detected power is

$$P_0 = \langle P\rangle = \iint_A \langle I\rangle\, d^2r = \langle I\rangle A.$$

There are two noise sources by the detection: noise from the detector, and noise inherent in the received power P. The detector noise is in the IR-spectral region usually thermal noise and can be represented by a noise-equivalent power NEP. The noise inherent in the received power may be set equal to σ_P, the standard deviation of P. Since these two noise sources are independent, the total noise becomes

$$N = \sqrt{(NEP)^2 + \sigma_P^2}.$$

Thus, the signal-to-noise ratio by direct detection can be written

$$(S/N)_d = \frac{P_0}{\sqrt{(NEP)^2 + \sigma_P^2}}. \quad (1)$$

We shall now calculate the variance $\sigma_P^2 = \langle P^2\rangle - \langle P\rangle^2$ in terms of the statistical properties of the incident light.

$$\langle P^2\rangle = \left\langle \iint_A I(\mathbf{r}_1)\, d^2r_1 \iint_A I(\mathbf{r}_2)\, d^2r_2 \right\rangle$$
$$= \iint_A d^2r_1 \iint_A \langle I_1 I_2\rangle\, d^2r_2,$$

where $I_1 = I(\mathbf{r}_1)$ and $I_2 = I(\mathbf{r}_2)$.

We introduce the normalized intensity covariance by

$$C_I = \frac{\langle I_1 I_2\rangle - \langle I\rangle^2}{\sigma_I^2}$$

and assume that C_I only depends on $\varrho = |\mathbf{r}_1 - \mathbf{r}_2|$, i.e., $C_I = C_I(\varrho)$. Then,

$$\langle P^2\rangle = P_0^2 + \sigma_I^2 \iint_A d^2r_1 \iint_A C_I(\varrho)\, d^2r_2. \quad (2)$$

Now, the backscattered field is to a good approximation a complex Gaussian process, i.e., the amplitude is Rayleigh distributed, the phase is uniformly distributed, and the intensity has a negative exponential distribution. This is true when the field is backscattered from an incoherent target (e.g., atmospheric aerosol particles) and in the absence of atmospheric turbulence (Goodman, 1976). But it is also true both at sufficiently weak turbulence and strong turbulence (Lee et al., 1976; Clifford et al., 1978; Pincus et al., 1978). For intermediate values of the turbulence, Pincus et al. (1978) experimentally found a small deviation (at most about 12%) from the unity value of the standard deviation of the intensity predicted by the complex Gaussian process model of the backscattered field. Thus, in the following we assume the backscattered field to be a complex Gaussian process.

Then (see Appendix A),

$$\sigma_I^2 = \langle I\rangle^2, \quad (3)$$

and

$$C_I(\varrho) = \gamma^2(\varrho), \quad (4)$$

where

$$\gamma(\varrho) = \frac{\langle E_1 E_2^*\rangle}{\langle |E|^2\rangle}$$

is the coherence function for the received field E. As before, $E_1 = E(\mathbf{r}_1)$, etc.

Substitution of Eqs. (3) and (4) into Eq. (2) gives

$$\langle P^2\rangle = P_0^2 + \langle I\rangle^2 \iint_A d^2r_1 \iint_A \gamma^2(\varrho)\, d^2r_2.$$

It is convenient to define

$$m = A^2 \left(\iint_A d^2r_1 \iint_A \gamma^2(\varrho)\, d^2r_2 \right)^{-1}. \quad (5)$$

Physically, m can be interpreted as the number of correlation cells over the aperture area (Goodman, 1965*a*).

We obtain

$$\langle P^2\rangle = P_0^2\left(1 + \frac{1}{m}\right),$$
$$\sigma_P^2 = \frac{P_0^2}{m},$$

and finally from Eq. (1)

$$(S/N)_d = \frac{P_0}{\sqrt{(NEP)^2 + \dfrac{P_0^2}{m}}}. \quad (6)$$

The expression for m [Eq. (5)] can be simplified in the following way: Introduce the pupil function

$$D(\mathbf{r}) = \begin{cases} 1 & \text{when } r \le r_0, \\ 0 & \text{when } r > r_0, \end{cases}$$

where r_0 is the radius of the circular area A. Furthermore, change the variables of integration by

$$\boldsymbol{\rho} = \mathbf{r}_1 - \mathbf{r}_2$$
$$2\boldsymbol{\varkappa} = \mathbf{r}_1 + \mathbf{r}_2.$$

Then

$$m = A^2\left(\iint_{-\infty}^{+\infty} \gamma^2(\varrho)\, d^2\varrho \iint_{-\infty}^{+\infty} D(\boldsymbol{\varkappa} + \tfrac{1}{2}\boldsymbol{\rho})\, D(\boldsymbol{\varkappa} - \tfrac{1}{2}\boldsymbol{\rho})\, d^2\varkappa \right)^{-1}$$
$$= A^2\left(\iint_{-\infty}^{+\infty} \gamma^2(\varrho)\, M(\varrho)\, d^2\varrho \right)^{-1},$$

where

$$M(\varrho) = \iint_{-\infty}^{+\infty} D(\boldsymbol{\varkappa} + \tfrac{1}{2}\boldsymbol{\rho})\, D(\boldsymbol{\varkappa} - \tfrac{1}{2}\boldsymbol{\rho})\, d^2\varkappa.$$

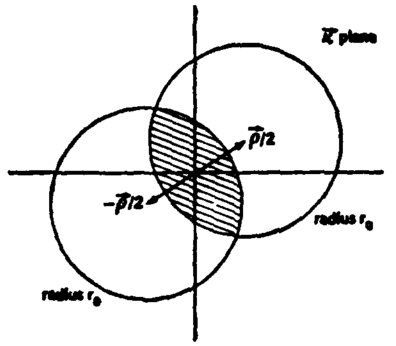

Fig. 1

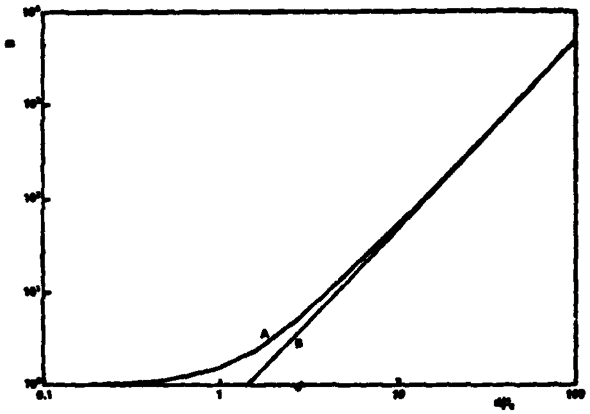

Fig. 2. (A) The number, m, of correlation cells in the receiver aperture as a function of d/ϱ_c, where d is the aperture diameter, and ϱ_c is the lateral coherence length of the incident light. (B) The asymptote $\frac{1}{2}(d/\varrho_c)^2$.

It is not difficult to realize that $M(\varrho)$ = the shaded area in Fig. 1. By simple geometrical arguments, we find

$$M(\varrho)=\begin{cases}0 & \text{for } \varrho>2r_0,\\ 2r_0^2\arccos\dfrac{\varrho}{2r_0}-\varrho\sqrt{r_0^2-\dfrac{\varrho^2}{4}} & \text{for } \varrho\leqslant 2r_0.\end{cases}$$

Thus,

$$m=\frac{A^2}{2\pi}\left(\int_0^{2r_0}\gamma^2(\varrho)M(\varrho)\varrho\,d\varrho\right)^{-1}$$
$$=\frac{\pi}{16}\left(\int_0^1\varrho(\arccos\varrho-\varrho\sqrt{1-\varrho^2})\gamma^2(\varrho d)\,d\varrho\right)^{-1}, \quad (7)$$

where d is the diameter $2r_0$.

The coherence function $\gamma(\varrho)$ appearing in the expression for m has recently been studied by Clifford et al. (1978) and Yura (1978). They consider a lidar system with backscattering from aerosol particles and take into account near-field effects and atmospheric turbulence. The field distribution E_T over the lidar transmitter aperture is assumed to be Gaussian (this corresponds to the lowest-order mode of a laser with confocal mirrors):

$$E_T=E_0\exp\left[-\frac{r^2}{2}\left(\frac{1}{a^2}+\frac{ik}{f}\right)\right]. \quad (8)$$

Here, E_0 is the field at the aperture center, r is the distance from the center, a is the $1/e$ intensity radius, $k=2\pi/\lambda$, and f is the focal length. The coherence function becomes

$$\gamma(\varrho)=\exp\left[-\left(\frac{\varrho}{\varrho_c}\right)^2\right], \quad (9)$$

with the coherence length ϱ_c given by

$$\varrho_c^2=\left(\frac{2z}{k}\right)^2\left[a^2\left(1-\frac{z}{f}\right)^2+\left(\frac{z}{ka}\right)^2+\left(\frac{z}{0.22k\varrho_0}\right)^2\right]^{-1}. \quad (10)$$

Here, z is the distance to the scattering region, and ϱ_0 is the coherence length for a spherical wave which has propagated a distance z through a turbulent atmosphere,

$$\varrho_0=\left[1.45k^2\int_0^z C_n^2(s)\left(\frac{s}{z}\right)^{5/3}ds\right]^{-3/5}.$$

C_n is the index-of-refraction structure function and is a measure of the fluctuations in the index-of-refraction.

Substitution of Eq. (9) into Eq. (7) gives

$$m=\frac{\pi}{16}\left(\int_0^1\varrho(\arccos\varrho-\varrho\sqrt{1-\varrho^2})\exp\left[-2\left(\frac{\varrho d}{\varrho_c}\right)^2\right]d\varrho\right)^{-1}. \quad (11)$$

This function $m=m(d/\varrho_c)$ was calculated numerically and is shown in Fig. 2. Asymptotically m tends to $\frac{1}{2}(d/\varrho_c)^2$, which can be found by approximating

$$\varrho(\arccos\varrho-\varrho\sqrt{1-\varrho^2})=\tfrac{1}{2}\pi\varrho$$

in the integrand in Eq. (11). The integral can then be solved analytically.

To demonstrate the effects of the speckle noise on a direct-detection lidar, we now consider the following example of lidar data:

$NEP=10^{-7}$ watt
Transmitted pulse energy = 1 J
Optical efficiency = 0.3
Backscattering coefficient = 10^{-7} m^{-1} sr^{-1}
Atmospheric attenuation coefficient = 0.1 km^{-1}
Receiver diameter = 0.5 m
Radius of transmitted beam (the quantity a in Eq. (8)) = 0.1 m
$f=\infty$ (collimated beam)
Wavelength = 10 μm.

Figs. 3–5 show results of numerical calculations obtained from these assumptions. Fig. 3 shows the coherence length ϱ_c as a function of the distance z calculated from Eq. (10).

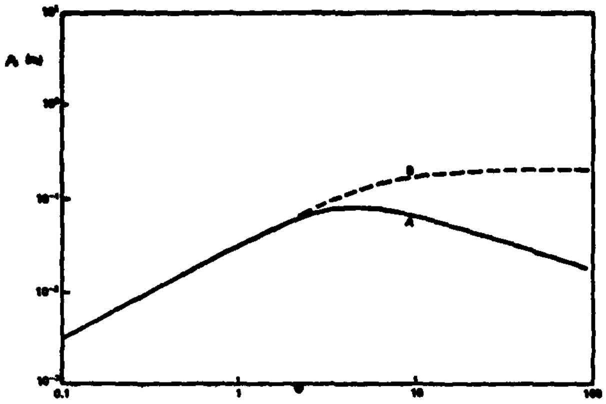

Fig. 3. (A) The lateral coherence length, ρ_c, as a function of the distance, z, for a case with atmospheric turbulence ($C_n^2 = 10^{-14}\ \mathrm{m}^{-2/3}$). (B) Same for a case with no atmospheric turbulence ($C_n^2 = 0$).

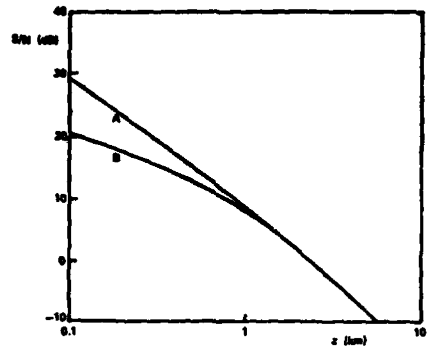

Fig. 5. The signal-to-noise ratio, S/N, as a function of the distance, z. (A) Without regard to the speckle noise (P_0/NEP). (B) With regard to the speckle noise. The atmospheric turbulence does not affect the results of this example, at least not as long as $C_n^2 \lesssim 10^{-14}\ \mathrm{m}^{-2/3}$.

Results are shown both for the case where there is no turbulence ($C_n^2=0$) along the propagation path, and where there is turbulence ($C_n^2=10^{-14}\ m^{-2/3}$).

Fig. 4 shows the corresponding values of the number m of correlation cells calculated from Eq. (11).

Finally, Fig. 5 shows $(S/N)_d$ calculated from Eq. (6). The atmospheric turbulence does not affect the result, at least not as long as $C_n^2 \lesssim 10^{-14}\ m^{-2/3}$. In the figure is also shown P_0/NEP, which is the value of the signal-to-noise ratio without regard to the speckle noise.

It is seen from Fig. 5 that there is a considerable decrease in the signal-to-noise ratio at short distances from the lidar due to the speckle noise. In a real direct-detection lidar system this decrease can be smaller for a number of reasons.

One reason is that in addition to the spatial averaging just discussed, there may also be an averaging in the frequency domain. The necessary wave-number change in order to get a new independent sample of the backscattered field is $1/l$, where $l = ct_p$ is the pulse length, $c =$ velocity of light, and t_p the pulse width (Marshall & Hitschfeld, 1953; Wallace, 1953). Assume that the spectral width of the backscattered light is $\Delta k > 1/l$ wavenumbers. Then the number m_k of independent samples over which averaging occurs, is

$$m_k = l\Delta k. \tag{12}$$

In calculating $(S/N)_d$ from Eq. (6), one has to replace m by mm_k.

The spectral width of the backscattered light may either be defined by the initial laser line width or by the Doppler broadening. If the Doppler broadening dominates, $\Delta k = v/\lambda c$, and

$$m_k = \frac{t_p}{t_s},$$

where $t_s = \lambda/v$ is the time scale for the speckle noise due to the turbulent velocity v in the scattering volume.

As an example, for a NASA direct-detection DIAL under development (Stewart & Bufton, 1978), the laser line width will be 0.033 cm^{-1} and the pulse width 50 ns. Then, from Eq. (12), $m_k = 50$.

The speckle noise can of course also be reduced by averaging in the time domain. If the integration time or gate time $t_g > \mathrm{Min}(t_p, t_s)$, then there will be an averaging over m_t independent samples, where

$$m_t = \frac{t_g}{\mathrm{Min}(t_p, t_s)}.$$

A further reason for the signal-to-noise decrease to be smaller than shown in Fig. 5 is that the spatial averaging over the receiver aperture can be more effective than calculated. In the derivation of Eq. (10) it was assumed that the laser was in single-mode operation. However, for a

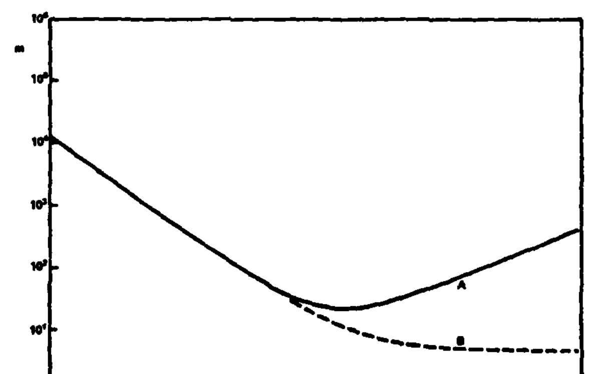

Fig. 4. (A) The number, m, of correlation cells in the receiver aperture as a function of the distance, z, for the case with atmospheric turbulence ($C_n^2 = 10^{-14}\ \mathrm{m}^{-2/3}$). (B) Same for the case with no atmospheric turbulence ($C_n^2 = 0$).

laser in multi-mode operation, the beam divergence can be larger. This results in a smaller coherence length ϱ_c, a larger number of correlation cells over the receiver aperture, and a more effective spatial averaging. If atmospheric turbulence effects are negligible, the coherence length can be estimated from the van Cittert-Zernike theorem (Born & Wolf, 1965, p. 508). Approximately, in the far field the coherence length ϱ_c becomes λ/θ, where θ is the beam divergence.

Finally, I want to point out, that in this analysis atmospheric turbulence has been considered only with regard to its effects on the coherence length ϱ_c. But the turbulence also gives rise to extra noise (scintillations) in the signal. This effect has not been taken into account. However, it is probably often a small effect, as mentioned before.

3. HETERODYNE DETECTION

In the same way as Eq. (1) was used for a direct-detection system, we set for the signal-to-noise ratio of a lidar with heterodyne detection

$$(S/N)_h = \frac{P_0}{\sqrt{(NEP)^2 + \sigma_P^2}}. \tag{13}$$

P_0 is now no longer the intensity integrated over the total receiver area, but instead (Goodman, 1965b)

$$P_0 = \langle I \rangle A_{eff},$$

with the effective area

$$A_{eff} = \frac{1}{A} \iint_A d^2 r_1 \iint_A \gamma d^2 r_2 \leqslant A.$$

There is no spatial averaging of the intensity fluctuations. Therefore, assuming, as before, a negative exponential distribution for the intensity,

$$\sigma_P = P_0.$$

The NEP now is the noise-equivalent power due to the local oscillator shot noise,

$$NEP = \frac{h\nu B}{\eta}.$$

Here h is Planck's constant, ν the frequency, B the IF bandwidth, and η the quantum efficiency. Often, $P_0 \gg NEP$, and then $(S/N)_h \approx 1$.

Usually one chooses $B = 1/t_p$. Since the necessary frequency change to get an independent sample of the backscattered field also is $1/t_p$, we see that there can usually be no averaging of the speckle noise in the frequency domain.

However, if $P_0 > NEP$, we can in fact increase the signal-to-noise ratio by choosing $B > 1/t_p$. The variance σ_P^2 will be decreased by the factor $m_h = Bt_p$. From Eq. (13) we get

$$(S/N)_h = \frac{P_0}{\sqrt{\left(\frac{h\nu B}{\eta}\right)^2 + \frac{P_0^2}{Bt_p}}}.$$

It is easy to show that the highest, in this way, attainable signal-to-noise ratio is

$$(S/N)_h = 0.89[(S/N)_0]^{1/3},$$

where

$$(S/N)_0 = \frac{P_0 \eta t_p}{h\nu}$$

is the signal-to-noise ratio in the absence of speckle noise and with $B = 1/t_p$.

The corresponding optimal value of the IF bandwidth is

$$B = \frac{0.79[(S/N)_0]^{2/3}}{t_p}. \tag{14}$$

A necessary condition for these results is of course that the bandwidth of the backscattered light is $\geqslant B$.

4. COMPARISON BETWEEN DIRECT DETECTION AND HETERODYNE DETECTION

It may be of interest to try to compare a lidar system with direct detection and one with heterodyne detection. For simplicity, let us assume that the two systems are identical (same pulse energy, optical efficiency, etc.) except for the detection technique. The noise-equivalent power for the direct-detection system is $(NEP)_d$ and is determined by the detector thermal noise. For the heterodyne case it is $(NEP)_h$, the shot noise induced by the local oscillator. Typically, at IR wavelengths $(NEP)_d$ is larger than $(NEP)_h$ by several orders of magnitude. However, in the direct-detection system the speckle noise is effectively averaged out in the spatial and frequency domains, and it may often be possible to obtain

$$(S/N)_d \approx \frac{P_0}{(NEP)_d}.$$

In the heterodyne case, on the other hand, we get, as was shown above, at most

$$(S/N)_h = 0.89\left(\frac{P_0}{(NEP)_h}\right)^{1/3}.$$

Therefore, the direct-detection technique gives a higher signal-to-noise ratio, at least as long as

$$\frac{P_0}{(NEP)_d} > 0.89\left(\frac{P_0}{(NEP)_h}\right)^{1/3},$$

or

$$P_0 > 0.85\frac{(NEP)_d^{3/2}}{(NEP)_h^{1/2}}.$$

If the IF bandwidth is not optimized in the sense of Eq. (14), but is $B = 1/t_p$, we get instead

$$P_0 > (NEP)_d.$$

In conclusion, the direct-detection system has the largest signal-to-noise ratio at short distances, whereas at longer distances the heterodyne-detection system is superior.

5. DIFFERENTIAL ABSORPTION

In this chapter we will consider the sensitivity limitations imposed by the speckle noise on differential-absorption measurements with a heterodyne-detection lidar.

To start with, let us set up the basic equations. The lidar equation can be written

$$P_0 = \frac{C\beta(z)}{z^2} \exp\left[-2\int_0^z (\sigma N_z + \alpha)\, dz\right], \tag{15}$$

where P_0 is the average received power, $\beta(z)$ is the backscattering coefficient at distance z, σ is the absorption cross section for the gas of interest, α is the attenuation coefficient due to all other constituents of the atmosphere, N_z is the gas concentration at distance z, and C is a constant.

It follows from Eq. (15) that the average gas concentration N_0 over the distance $z_1 - z_2$, i.e.,

$$N_0 = \frac{\int_{z_1}^{z_1} N_z\, dz}{z_1 - z_2},$$

is given by

$$N_0 = \frac{1}{\tau} \ln \frac{P_0(\lambda_2, z_1)\, P_0(\lambda_1, z_2)}{P_0(\lambda_1, z_1)\, P_0(\lambda_2, z_2)}, \tag{16}$$

where

$\tau \quad = 2[\sigma(\lambda_1) - \sigma(\lambda_2)](z_1 - z_2)$,

$\sigma(\lambda_1)$ = molecular absorption cross section on an absorption peak of the gas of interest,

$\sigma(\lambda_2)$ = molecular absorption cross section off the absorption peak, and

$P_0(\lambda_i, z_j)$ = averaged received power at wavelength λ_i and range z_j.

For the derivation of Eq. (16) it was assumed that λ_1 and λ_2 are so close to each other that the backscattering coefficient β and the attenuation coefficient α have the same values at the two wavelengths. This is a quite reasonable assumption for β, but may not always be true for α because of interference from other gases. However, by different compensation techniques these interference effects can be made negligible (Murray, 1978).

An estimate N of the gas concentration is obtained by

$$N = \frac{1}{\tau} \ln \frac{P(\lambda_2, z_1)\, P(\lambda_1, z_2)}{P(\lambda_1, z_1)\, P(\lambda_2, z_2)}. \tag{17}$$

We will consider the large-signal case, i.e., where $P_0 \gg NEP$. Then the uncertainty σ_N in the estimate N is due to fluctuations in the received power P. These fluctuations in turn are due to speckle noise, atmospheric turbulence, and variations in the backscattering coefficient β and the attenuation coefficient α.

The time scale for the turbulence-induced fluctuations is $\sqrt{\lambda z}/v$, where v is the wind velocity transverse to the propagation path (Clifford et al., 1978). Typically, $\sqrt{\lambda z}/v \approx 10^{-3} - 10^{-2}$ s. The time scale for the variations in backscattering coefficient β and the attenuation coefficient α is larger than 10^{-3} s (Schotland, 1974). It is therefore strongly to recommend that the measurements on and off the absorption line be made within a time which is smaller than about 10^{-3} s, since then the fluctuations due to turbulence and variable aerosol structure cancel out in the power ratios in Eq. (17). The best is, of course, if the power measurements are made simultaneously (Derr et al., 1974; Stewart & Bufton, 1978).

The wavelength dependence of the turbulence effects in Eq. (17) is negligible. The characteristic transverse length for correlation of the intensity fluctuations is $\sqrt{\lambda z}$ (Fante, 1975). Furthermore, in a differential-absorption measurement the fractional wavelength shift typically is about 0.2%. Therefore, the power scintillations at λ_1 and λ_2 can be considered to be completely correlated, and they cancel out in the power ratios in Eq. (17).

In the following we assume that the measurements on and off the absorption line are made effectively simultaneously. The only contribution to the uncertainty σ_N then comes from the speckle noise. The four samples $P(\lambda_2, z_1)$, $P(\lambda_1, z_2)$, $P(\lambda_1, z_1)$, and $P(\lambda_2, z_2)$ are certainly independent with regard to the speckle noise. This follows since an independent sample of the speckle noise is obtained if $|z_1 - z_2| > l$ or $|\lambda_1 - \lambda_2| > \lambda^2/l$, where $l = ct_p$ is the pulse length.

From Eq. (17) we get

$$\mathrm{Var}(N) = \frac{1}{\tau^2}\{\mathrm{Var}[\ln P(\lambda_2, z_1)] + \mathrm{Var}[\ln P(\lambda_1, z_2)] + \mathrm{Var}[\ln P(\lambda_1, z_1)] + \mathrm{Var}[\ln P(\lambda_2, z_2)]\},$$

where $\mathrm{Var}(\cdot)$ denotes the variance. The probability distribution function for P is

$$p(P) = \frac{1}{P_0} \exp\left(-\frac{P}{P_0}\right),$$

where P_0, as before, denotes the mean value. We obtain (Gradshteyn & Ryzhik, 1965, 4.331 and 4.335):

$$\begin{aligned}\mathrm{Var}(\ln P) &= \langle \ln^2 P\rangle - \langle \ln P\rangle^2 \\ &= \int_0^\infty p(P) \ln^2 P\, dP - \left(\int_0^\infty p(P) \ln P\, dP\right)^2 \\ &= \pi^{2/6}.\end{aligned}$$

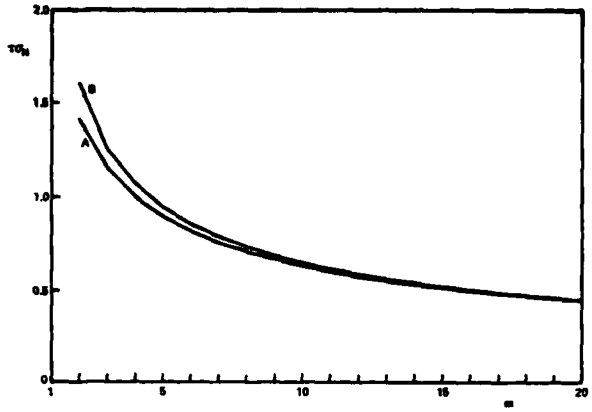

Fig. 6. The standard deviation, $\tau\sigma_N$, of the estimate of the gas concentration as a function of the number, m, of independent samples. The function is calculated from (A) an approximate expression (Eq. (19)), and from (B) an exact expression (Eq. 20)).

Thus,

$$\mathrm{Var}\,(N)=\frac{2\pi^2}{3\tau^2},$$

and

$$\sigma_N=\sqrt{\mathrm{Var}\,(N)}=\frac{2\pi}{\tau\sqrt{6}}=\frac{2.6}{\tau}.$$

In a sensitivity analysis it is reasonable to set the minimum detectable gas concentration, $N_{\min}$, equal to the standard deviation σ_N. Thus, $N_{\min}=2.6/\tau$. If we average over n pulses, the standard deviation of course instead becomes $\sigma_N/\sqrt{n}$.

Finally, it can be of some interest to consider the case where there is averaging of the speckle noise in the frequency domain, as discussed in Ch. 3. Then we have instead of Eq. (17):

$$N=\frac{1}{\tau}\,\frac{Q(\lambda_2,z_1)\,Q(\lambda_1,z_2)}{Q(\lambda_1,z_1)\,Q(\lambda_2,z_2)},\tag{18}$$

with

$$Q(\lambda_i,z_j)=\frac{1}{m}\sum_{1}^{m}P(\lambda_i,z_j),$$

and m is the number of independent samples of P over which averaging takes place. It follows that

$$\mathrm{Var}\,(N)=\frac{1}{\tau^2}\{\mathrm{Var}\,[\ln Q(\lambda_2,z_1)]+\mathrm{Var}\,[\ln Q(\lambda_1,z_2)] + \mathrm{Var}\,[\ln Q(\lambda_1,z_1)]+\mathrm{Var}\,[\ln Q(\lambda_2,z_2)]\}.$$

Now, if m is sufficiently large, then the probability distribution for Q is sufficiently narrow, and we can approximate (Bevington, 1969, p. 59):

$$\mathrm{Var}\,(\ln Q)=\left(\frac{d(\ln Q)}{dQ}\right)^2_{Q=Q_0}\mathrm{Var}\,(Q)=\frac{\mathrm{Var}\,(Q)}{Q_0^2},$$

where $Q_0=P_0$ is the mean value of Q. But

$$\mathrm{Var}\,(Q)=\frac{\mathrm{Var}\,(P)}{m}=\frac{P_0^2}{m}.$$

Thus,

$$\mathrm{Var}\,(\ln Q)=\frac{1}{m},$$

and

$$\mathrm{Var}\,(N)=\frac{4}{\tau^2 m},$$

$$\sigma_N=\frac{2}{\tau\sqrt{m}}.\tag{19}$$

It is possible to show (see Appendix B) that for a general value of $m>1$, we instead get

$$\sigma_N=\frac{2}{\tau}\sqrt{\frac{\pi^2}{6}-\sum_{\nu=1}^{m-1}\frac{1}{\nu^2}}.\tag{20}$$

The standard deviation calculated from Eqs. (19) and (20) is shown as function of m in Fig. 6.

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APPENDIX A

RELATION BETWEEN INTENSITY COVARIANCE AND COHERENCE FUNCTION

The field E is assumed to be a complex Gaussian process, i.e.,

$$E = |E|e^{i\phi} = x + iy,$$

where $x = |E|\cos\phi$ and $y = |E|\sin\phi$ are independent, zero mean Gaussian random variables with variances σ^2. The probability distribution is

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{x^2}{2\sigma^2}\right).$$

The intensity is

$$I = |E|^2 = x^2 + y^2.$$

Thus

$$\langle I\rangle = \langle x^2\rangle + \langle y^2\rangle = 2\sigma^2,$$

$$\langle I^2\rangle = \langle x^4\rangle + \langle y^4\rangle + 2\langle x^2\rangle\langle y^2\rangle = 8\sigma^4.$$

It follows that

$$\sigma_I^2 = \langle I^2\rangle - \langle I\rangle^2 = \langle I\rangle^2,$$

which is Eq. (3).

We shall now relate

$$C_I = \frac{\langle I_1 I_2\rangle - \langle I\rangle^2}{\sigma_I^2} \quad \text{to } \gamma = \frac{\langle E_1 E_2^*\rangle}{\langle |E|^2\rangle}.$$

$$\begin{aligned}\langle I_1 I_2\rangle &= \langle (x_1^2 + y_1^2)(x_2^2 + y_2^2)\rangle \\ &= \langle x_1^2 x_2^2\rangle + \langle x_1^2\rangle\langle y_2^2\rangle + \langle x_2^2\rangle\langle y_1^2\rangle + \langle y_1^2 y_2^2\rangle \\ &= 2(\langle x_1^2 x_2^2\rangle + \sigma^4),\end{aligned}$$

where we have used that $\langle x^2\rangle = \langle y^2\rangle = \sigma^2$ and, by symmetry, $\langle y_1^2 y_2^2\rangle = \langle x_1^2 x_2^2\rangle$. $\langle x_1^2 x_2^2\rangle$ can be expressed in $\langle x_1 x_2\rangle$ in the following way (Burdic, 1968, p. 277):

Introduce a new variable z by $x_2 = cx_1 + z$, where the constant c is defined by $c = \langle x_1 x_2\rangle/\sigma^2$. It is then easy to show that x_1 and z are uncorrelated, i.e., $\langle x_1 z\rangle = 0$. But since x_1 and z also are Gaussian, it follows that they also are independent. We then find that

$$\langle x_1^2 x_2^2\rangle = \langle x_1^2(cx_1 + z)^2\rangle = 3\langle x_1 x_2\rangle^2 + \sigma^2\langle z^2\rangle.$$

Furthermore,

$$\langle z^2\rangle = \langle (x_2 - cx_1)^2\rangle = \sigma^2 - \frac{\langle x_1 x_2\rangle^2}{\sigma^2}.$$

Therefore,

$$\langle x_1^2 x_2^2\rangle = 2\langle x_1 x_2\rangle^2 + \sigma^4.$$

Now,

$$C_I = \frac{\langle I_1 I_2\rangle - \langle I\rangle^2}{\langle I\rangle^2} = \frac{\langle x_1 x_2\rangle^2}{\sigma^4},$$

$$\gamma = \frac{\langle E_1 E_2^*\rangle}{\langle |E|^2\rangle} = \frac{\langle x_1 x_2\rangle}{\sigma^2}.$$

Combination of the last two equations finally gives

$$C_I = \gamma^2,$$

which is Eq. (4).

APPENDIX B

DERIVATION OF A FORMULA FOR THE STANDARD DEVIATION OF ESTIMATED GAS CONCENTRATION

The basic equation is Eq. (18):

$$N = \frac{1}{\tau} \frac{Q(\lambda_2, z_1) Q(\lambda_1, z_2)}{Q(\lambda_1, z_1) Q(\lambda_2, z_2)}, \qquad \text{(B-1)}$$

with

$$Q(\lambda_i, z_j) = \frac{1}{m} \sum_1^m P(\lambda_i, z_j).$$

The probability distribution $p_P(P)$ for the received power P is the negative exponential,

$$p_P(P) = \frac{1}{P_0} \exp\left(-\frac{P}{P_0}\right).$$

From *this, we can derive the probability* distribution $p_N(N)$ and the standard deviation σ_N for the estimate N defined in Eq. (B-1).

First we note that the probability distribution function $p_Q(Q)$ for Q is the gamma distribution (Marshall & Hitschfeld, 1953; Wallace, 1953):

$$p_Q(Q) = \frac{m^m}{Q_0^m (m-1)!} Q^{m-1} \exp\left(-\frac{mQ}{Q_0}\right),$$

where $Q_0 = P_0$ is the mean value of Q and P.

Next, introduce random variables X_1, X_2, and X_3 by

$$X_1 = \frac{Q(\lambda_2, z_1)}{Q(\lambda_1, z_1)}, \quad X_2 = \frac{Q(\lambda_1, z_2)}{Q(\lambda_2, z_2)}, \quad X_3 = X_1 X_2,$$

so that

$$N = \frac{1}{\tau} \ln X_3.$$

Now, in general, if $x \geq 0$ and $y \geq 0$ are independent random variables with probability density functions $p_x(x)$ and $p_y(y)$, and if $z = x/y$, then the probability density function for z is

$$p_z(z) = \int_0^\infty y\, p_x(yz)\, p_y(y)\, dy.$$

If, instead, $z = xy$, then

$$p_z(z) = \int_0^\infty \frac{1}{y} p_x\left(\frac{z}{y}\right) p_y(y)\, dy.$$

With these formulas, the probability density functions for X_1, X_2, and X_3 can be derived. Furthermore, the probability density functions $p_N(N)$ and $p_{X_3}(X_3)$ for N and X_3 are related by

$$p_N(N) = \tau e^{\tau N} p_{X_3}(e^{\tau N}).$$

In this way $p_N(N)$ can be calculated. The result is

$$p_N(N) = \left[\frac{(2m-1)!}{[(m-1)!]^2}\right]^2 \tau \exp[m\tau(N - N_0)] \times \int_0^\infty \frac{t^{2m-1}\, dt}{(t+1)^{2m} \{t + \exp[\tau(N - N_0)]\}^{2m}}.$$

Here $N = N_0$ is the mean value of N_0, and is defined in Eq. (16). The kth moment of N becomes

$$\begin{aligned} \langle N^k \rangle &= \int_{-\infty}^{+\infty} N^k p_N(N)\, dN \\ &= \left[\frac{(2m-1)!}{[(m-1)!]^2}\right]^2 \tau \int_0^\infty \frac{t^{2m-1}}{(t+1)^{2m}}\, dt \\ &\quad \times \int_{-\infty}^{+\infty} \frac{N^k \exp[m\tau(N - N_0)]}{\{t + \exp[\tau(N - N_0)]\}^{2m}}\, dN. \end{aligned}$$

Change of variable of integration in the last integral by

$$s = \frac{\exp[\tau(N - N_0)]}{t}$$

gives

$$\langle N_k \rangle = \left[\frac{(2m-1)!}{[(m-1)!]^2}\right]^2 \frac{1}{\tau^k} \int_0^\infty \frac{t^{m-1}}{(t+1)^{2m}}\, dt \times \int_0^\infty \frac{\{\ln[st \exp(\tau N_0)]\}^k s^{m-1}}{(s+1)^{2m}}\, ds.$$

If we furthermore utilize that

$$\begin{aligned} \{\ln[st \exp(\tau N_0)]\}^k &= (\ln s + \ln t + \tau N_0)^k \\ &= \sum_{\nu=0}^{k} \sum_{\mu=0}^{\nu} \binom{k}{\nu} \binom{\nu}{\mu} (\ln s)^{k-\nu} (\ln t)^{\mu} (\tau N_0)^{\nu-\mu}, \end{aligned}$$

then

$$\langle N^k \rangle = \frac{1}{\tau^k} \left[\frac{(2m-1)!}{[(m-1)!]^2}\right]^2 \sum_{\nu=0}^{k} \sum_{\mu=0}^{\nu} \binom{k}{\nu} \binom{\nu}{\mu} A_\mu A_{k-\nu} (\tau N_0)^{\nu-\mu},$$

where

$$A_\mu = \int_0^\infty \frac{t^{m-1} (\ln t)^\mu}{(t+1)^{2m}}\, dt. \qquad \text{(B-2)}$$

The problem of evaluating $\langle N \rangle$ is thus reduced to evaluating integrals of the type A_μ in Eq. (B-2). It can be shown that

$$A_\mu = 0 \quad \text{when } \mu \text{ is odd},$$

$$A_0 = \frac{[(m-1)!]^2}{(2m-1)!},$$

$$A_2 = 2 \frac{[(m-1)!]^2}{(2m-1)!} \left(\frac{\pi^2}{6} - \sum_{\nu=1}^{m-1} \frac{1}{\nu^2}\right).$$

With these results it follows that

$$\langle N\rangle = N_0,$$

$$\langle N^2\rangle = N_0^2 + \frac{4}{\tau^2}\left(\frac{\pi^2}{6} - \sum_{\nu=1}^{m-1}\frac{1}{\nu^2}\right),$$

and

$$\sigma_N = \frac{2}{\tau}\sqrt{\frac{\pi^2}{6} - \sum_{\nu=1}^{m-1}\frac{1}{\nu^2}} \qquad \text{(B-3)}$$

which is the desired result. For this to be consistent with Eq. (19), it has to be shown that σ_N given by Eq. (B-3) tends to $2/(\tau\sqrt{m})$ when $m \to \infty$. To this end, we first note that

$$\sum_{\nu=1}^{\infty}\frac{1}{\nu^2} = \frac{\pi^2}{6},$$

and therefore

$$\frac{\pi^2}{6} - \sum_{\nu=1}^{m-1}\frac{1}{\nu^2} = \sum_{\nu=m}^{\infty}\frac{1}{\nu^2} = \sum_{\nu=0}^{\infty}\frac{1}{(\nu+m)^2}.$$

But (Gradshteyn & Ryzhik, 1965, 8.360 and 8.363)

$$\sum_{\nu=0}^{\infty}\frac{1}{(\nu+x)^2} = \frac{d^2}{dx^2}\ln\Gamma(x),$$

where $\Gamma(x)$ is the gamma function.

Also,

$$\ln\Gamma(x) = x\ln x \quad \text{when} \quad x \to \infty.$$

Therefore,

$$\sum_{\nu=0}^{\infty}\frac{1}{(\nu+x)^2} = \frac{1}{x} \quad \text{when } x \to \infty,$$

$$\frac{\pi^2}{6} - \sum_{\nu=1}^{m-1}\frac{1}{\nu^2} = \frac{1}{m} \quad \text{when } m \to \infty,$$

and

$$\sigma_N = \frac{2}{\tau\sqrt{m}} \quad \text{when } m \to \infty.$$

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